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# About the self-dual Chern–Simons system and Toda field theories on the noncommutative plane

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## Abstract

The relation of the noncommutative self-dual Chern–Simons (NCSDCS) system to the noncommutative generalizations of Toda and of affine Toda field theories is investigated more deeply. This paper continues the programme initiated in (Cabrera-Carnero I 2005 *J. High Energy Phys.* JHEP10(2005)071), where it was presented how it is possible to define Toda field theories through second order differential equation systems starting from the NCSDCS system. Here we show that using the connection of the NCSDCS to the noncommutative chiral model, exact solutions of the Toda field theories can also be constructed by means of the noncommutative extension of the uniton method proposed in (Lee K-M 2004 *J. High Energy Phys.* JHEP08(2004)054) by Lee. Particularly some specific solutions of the nc Liouville model are explicitly constructed.

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## 1. Introduction

Inside the context of noncommutative field theories (NCFT), noncommutative (nc) extensions of two-dimensional integrable field theories have been investigated [1–12] in the last few years. Particularly in [12] nc extensions of Toda and affine Toda theories were proposed. The nc extensions of Toda field theories were constructed in [12] starting from a nc zero-curvature condition for algebra-valued potentials introduced in [5]. Expressing the gauge potentials in a particular way this condition can be reduced to the nc Leznov–Saveliev equation, which as shown in [12] can be regarded as the equation of motion of a constrained nc Wess–Zumino–Novikov–Witten model (WZNW<sub>\*</sub>). In this sense the extensions of Abelian and Abelian affine Toda theories presented in [12] have the advantage of possessing an infinite number of conserved charges and of being represented by second order differential equations for  $N$  fields. The corresponding action principles were also presented in [12].

On the other hand in [9] was considered the nc extension to the nc plane of the Dunne–Jackiw–Pi–Trugenberg (DJPT) model [13] of a  $U(N)$  Chern–Simons gauge theory coupled to a nonrelativistic complex adjoint matter field. The lowest energy solutions of this model satisfy a nc extension of the self-dual Chern–Simons equations. Through a proposed ansatz, nc generalizations of  $U(N)$  Toda and affine Toda theories were constructed from these nc self-dual equations. The generalizations of Toda theories proposed were expressed as systems of first order differential equations for  $2N - 1$  fields which could not be reduced to coupled second order equations in general. The advantage of defining the Toda field theories in this way was the possibility of constructing exact solutions since the self-dual equations for Chern–Simons solitons on nc space can be related to the equation of the  $U(N)$  nc chiral model, which apparently can also be solved by the Uhlenbeck’s unition method [16] as was suggested in [9]. In [12] was shown that the NCSDCS system can be reduced to the nc Leznov–Saveliev equations using a different ansatz and in this way obtain the Toda field theories as second order differential equations. In this paper we would like to present the relation between the NCSDCS system and the nc Toda field theories in a more detailed way, essentially in connection with the construction of exact solutions that as we will see is still possible. In this sense this paper complements the results presented in the last section of [12].

This paper is organized as follows. In the first section we review the derivation of the nc extensions of Abelian and Abelian affine Toda field theories presented in [12]. In section 3 we present how the NCSDCS system can be transformed into the nc Leznov–Saveliev equations from where the nc extensions of Toda field theories were constructed in [12]. In this section we also show how the system of first order differential equations for  $2N - 1$  fields considered [9] as the nc extension of Toda field theories can be reduced to our nc extension of Toda theories, i.e. a system of second order differential equations for  $N$  fields. In section 4 the relation of the NCSDCS to the nc principal chiral model [9] is reviewed. The nc extension of the unition method proposed in [9] and the explicit construction of some solutions of our nc extension of Liouville model through this method is as well exposed in this section. The last section provides the conclusions.

## 2. Toda theories from WZNW $\star$

It is well known that Toda theories connected with finite simple Lie algebras, on the ordinary commutative case, can be regarded as constrained Wess–Zumino–Novikov–Witten (WZNW) models [17]<sup>1</sup>. By placing certain constraints on the chiral currents, the  $G$ -invariant WZNW model reduces to the appropriate Toda theory. Specifically, the Abelian Toda theories are connected with Abelian embeddings  $G_0 \subset G$ . In [12] we constructed nc extensions of Abelian and Abelian affine Toda theories applying this procedure to the nc extension of the WZNW model (WZNW $\star$ ). Here we will briefly review our results.

As usual NCFT [22] can be constructed from a given field theory by replacing the product of fields by an associative  $\star$ -product. Considering that the noncommutative parameter  $\theta^{\mu\nu}$  is a constant antisymmetric tensor, the deformed product of functions is expressed through the Moyal product [23]:

$$\phi_1(x)\phi_2(x) \rightarrow \phi_1(x) \star \phi_2(x) = e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu^{x_1}\partial_\nu^{x_2}}\phi_1(x_1)\phi_2(x_2)|_{x_1=x_2=x}. \quad (1)$$

In the following we will refer to functions of operators in the noncommutative deformation by a  $\star$  sub-index, for example  $e_\star^\phi = \sum_{n=1}^{\infty} \frac{1}{n!}\phi_\star^n$  (the  $n$ -times star product of  $\phi$  is understood).

<sup>1</sup> Affine Toda theories can be as well regarded as constrained two-loop WZNW models [18].

Consider now the nc generalization of the WZNW model introduced in [20]

$$S_{\text{WZNW}_\star} = -\frac{k}{4\pi} \int_{\Sigma} d^2z \operatorname{Tr}(g^{-1} \star \partial g \star g^{-1} \star \bar{\partial} g) + \frac{k}{24\pi} \int_B d^3x \epsilon_{ijk} (g^{-1} \star \partial_i g \star g^{-1} \star \partial_j g \star g^{-1} \star \partial_k g). \tag{2}$$

Here  $B$  is a three-dimensional manifold whose boundary  $\partial B = \Sigma$ . We are considering that  $z = t + x$ ,  $\bar{z} = t - x$  and  $\partial = \frac{1}{2}(\frac{\partial}{\partial t} + \frac{\partial}{\partial x})$ ,  $\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial t} - \frac{\partial}{\partial x})$ . The coordinates  $z, \bar{z}$  or equivalently  $x, t$  are noncommutative, but the extended coordinate  $y$  on the manifold  $B$  remains commutative, i.e.  $[z, \bar{z}] = \theta$ ,  $[y, z] = [y, \bar{z}] = 0$ . The Euler–Lagrange equations of motion corresponding to (2) are

$$\bar{\partial} J = \partial \bar{J} = 0, \tag{3}$$

where  $J$  and  $\bar{J}$  represent the conserved chiral currents

$$J = g^{-1} \star \partial g, \quad \bar{J} = -\bar{\partial} g \star g^{-1}. \tag{4}$$

The fields  $\alpha_a$  parameterize the group element  $g \in G$  through  $g = e^{\alpha_a T_a}$ , where  $T_a$  are the generators of the corresponding algebra  $\mathcal{G}$ . As it was our interest to define the theories inside a  $G_0$  subgroup of  $G$ , the unwanted degrees of freedom that correspond to the tangent space  $G/G_0$  were eliminated implementing constraints upon specific components of the currents  $J, \bar{J}$ :

$$J_{\text{constr}} = j + \epsilon_-, \quad \bar{J}_{\text{constr}} = \bar{j} + \epsilon_+, \tag{5}$$

where  $\epsilon_{\pm}$  are constant elements of grade  $\pm 1$  with respect to a grading operator  $Q$ , i.e.  $[Q, \epsilon_{\pm}] = \pm \epsilon_{\pm}$  defined in the algebra  $\mathcal{G}$  and  $j, \bar{j}$  contain generators of grade zero and positive, and zero and negative, respectively. The grading operator  $Q$  decomposes the algebra  $\mathcal{G}$  in  $\mathbb{Z}$ -graded subspaces,

$$[Q, \mathcal{G}_i] = i\mathcal{G}_i, \quad [\mathcal{G}_i, \mathcal{G}_j] \in \mathcal{G}_{i+j}. \tag{6}$$

This means that the algebra  $\mathcal{G}$  can be represented as the direct sum  $\mathcal{G} = \bigoplus_i \mathcal{G}_i$  and that the subspaces  $\mathcal{G}_0, \mathcal{G}_>, \mathcal{G}_<$  are subalgebras of  $\mathcal{G}$ , composed of the Cartan and of the positive/negative steps generators, respectively. The algebra can then be written using the triangular decomposition  $\mathcal{G} = \mathcal{G}_< \oplus \mathcal{G}_0 \oplus \mathcal{G}_>$ . Denote the subgroup elements obtained through the  $\star$ -exponentiation of the generators of the corresponding subalgebras as  $N = e^{\mathcal{G}_<}, B = e^{\mathcal{G}_0}, M = e^{\mathcal{G}_>}$ . Then proposing a nc Gauss-like decomposition, an element  $g$  of the nc group  $G$  can be expressed as

$$g = N \star B \star M. \tag{7}$$

The reduced model is obtained then introducing (7) in (4) and giving the constant elements  $\epsilon_{\pm}$  of grade  $\pm 1$  responsible for constraining the currents in a general manner. As a result of the reduction process, the degrees of freedom in  $M, N$  are eliminated and the equations of motion of the constrained model are natural nc extensions of the Leznov–Saveliev equations of motion [19], namely

$$\begin{aligned} \bar{\partial}(B^{-1} \star \partial B) + [\epsilon_-, B^{-1} \star \epsilon_+ B]_{\star} &= 0, \\ \partial(\bar{\partial} B \star B^{-1}) - [\epsilon_+, B \epsilon_- \star B^{-1}]_{\star} &= 0. \end{aligned} \tag{8}$$

As shown in [5], the equations of motion (8) can be expressed as a generalized  $\star$ -zero-curvature condition

$$\bar{\partial} A - \partial \bar{A} + [A, \bar{A}]_{\star} = 0, \tag{9}$$

taking the potentials as  $A = -B\epsilon_- \star B^{-1}$  and  $\bar{A} = \epsilon_+ + \bar{\partial}B \star B^{-1}$ . This condition (9) implies the existence of an infinite amount of conserved charges [5]. For this reason in order to preserve the original integrability properties of the two-dimensional models (8) can be a reasonable starting point in order to construct nc analogues of Toda models.

Equations (8) are the Euler–Lagrange equations of motion of the action

$$S = S_{\text{WZNW}_\star}(B) + \frac{k}{2\pi} \int d^2z \text{Tr}(\epsilon_+ \star B\epsilon_- \star B^{-1}). \tag{10}$$

From (8) we constructed in [12] nc analogues of the  $GL(n, \mathbb{R})$  Abelian Toda theories taking the gradation operator as  $Q = \sum_{i=1}^{n-1} \frac{2\lambda_i \cdot H}{\alpha_i^2}$ , where  $H$  represents the Cartan subalgebra,  $\alpha_i$  is the  $i$ th simple root and  $\lambda_i$  is the  $i$ th fundamental weight that satisfies  $\frac{2\lambda_i \cdot \alpha_j}{\alpha_i^2} = \delta_{ij}$ . In this way it defines the subalgebra of grade zero as  $\mathcal{G}_0 = U(1)^n = \{I, h_i, i = 1, \dots, n - 1\}$ , where the Cartan generators are defined in the Chevalley basis as  $h_i = \frac{2\alpha_i \cdot H}{\alpha_i^2}$ . The zero-grade group element  $B$  is then expressed through the  $\star$ -exponentiation of the generators of the zero-grade subalgebra  $\mathcal{G}_0$ , i.e. the  $\mathcal{SL}(n)$  Cartan subalgebra plus the identity generator,

$$B = e_{\star}^{\sum_{i=1}^{n-1} \varphi_i h_i + \varphi_0 I}. \tag{11}$$

With the constant generators of grade  $\pm 1$  as

$$\epsilon_{\pm} = \sum_{i=1}^{n-1} \mu_i E_{\pm\alpha_i}, \tag{12}$$

where  $E_{\pm\alpha_i}$  are the step generators associated with the positive/negative simple roots of the algebra and  $\mu_i$  are constant parameters, the equations of motion that define the nc extension of Abelian Toda models are

$$\partial(\bar{\partial}(e_{\star}^{\phi_k}) \star e_{\star}^{-\phi_k}) = \mu_k^2 e_{\star}^{\phi_{k+1}} \star e_{\star}^{-\phi_k} - \mu_{k-1}^2 e_{\star}^{\phi_k} \star e_{\star}^{-\phi_{k-1}}, \tag{13}$$

a system of  $n$ -coupled equations ( $k = 1, \dots, n$ ) where we have changed variables to

$$\begin{aligned} \varphi_1 + \varphi_0 &= \phi_1, \\ -\varphi_k + \varphi_{k+1} + \varphi_0 &= \phi_{k+1}, \quad \text{for } k = 1 \text{ to } n - 2, \\ -\varphi_{n-1} + \varphi_0 &= \phi_n. \end{aligned} \tag{14}$$

Note that for the first and last equations  $\mu_0 = \mu_n = 0$  and  $\phi_0 = \phi_{n+1} = 0$ . A particular example of these field theories is the nc extension of the Liouville model:

$$\begin{aligned} \partial(\bar{\partial}(e_{\star}^{\phi_+}) \star e_{\star}^{-\phi_+}) &= \mu^2 e_{\star}^{\phi_-} \star e_{\star}^{-\phi_+}, \\ \partial(\bar{\partial}(e_{\star}^{\phi_-}) \star e_{\star}^{-\phi_-}) &= -\mu^2 e_{\star}^{\phi_+} \star e_{\star}^{-\phi_-}, \end{aligned} \tag{15}$$

with  $\phi_1 = \phi_+$  and  $\phi_2 = \phi_-$ . This system in the commutative limit will lead to a decoupled model of two fields: a free field and the usual Liouville field, i.e.

$$\partial\bar{\partial}\varphi_0 = 0 \quad \text{and} \quad \partial\bar{\partial}\varphi_1 = \mu^2 e^{-2\varphi_1}, \tag{16}$$

where  $\varphi_1 = \phi_+ - \phi_-$  and  $\varphi_0 = \phi_+ + \phi_-$ . The action, whose Euler–Lagrange equations of motion lead to (13), can be obtained from (10) with (11) and (12). It reads

$$S(\phi_1, \dots, \phi_n) = \sum_{k=1}^n S_{\text{WZNW}_\star}(e_{\star}^{\phi_k}) + \frac{k}{2\pi} \int d^2z \sum_{k=1}^{n-1} \mu_k^2 (e_{\star}^{\phi_{k+1}} \star e_{\star}^{-\phi_k}). \tag{17}$$

In the same way the nc extensions of  $\widetilde{GL}(n, \mathbb{R})$ <sup>2</sup> Abelian affine Toda theories can be constructed taking the gradation operator as  $Q = \sum_{i=1}^{n-1} \frac{2\lambda_i \cdot H^{(0)}}{\alpha_i^2} + nd$ , where  $d$  is the derivation

<sup>2</sup> We refer to the loop algebra, see [12] for more details.

generator whose coefficient is chosen such that this gradation ensures that the zero-grade subspace  $\mathcal{G}_0$  coincides with its counterpart on the corresponding Lie algebra  $\mathcal{SL}(n, \mathbb{R})$ , apart from the generator  $d$ . The major difference with the finite case is in the constant generators of grade  $\pm 1$  that include extra affine generators, say

$$\epsilon_{\pm} = \sum_{i=1}^{n-1} \mu_i E_{\pm\alpha_i}^{(0)} + m_0 E_{\mp\psi}^{(\pm 1)}, \tag{18}$$

here  $\psi$  is the highest root of  $\mathcal{G} = \mathcal{SL}(n, \mathbb{R})$  and  $m_0, \mu_i$  with  $i = 1, \dots, n - 1$  are constant parameters. Using again the nc extension of the Leznov–Saveliev equations of motion (8) the nc analogues of Abelian affine  $\widetilde{GL}(n, \mathbb{R})$  Toda theories are obtained

$$\partial(\bar{\partial}(e_{\star}^{\phi_k}) \star e_{\star}^{-\phi_k}) = \mu_k^2 e_{\star}^{\phi_{k+1}} \star e_{\star}^{-\phi_k} - \mu_{k-1}^2 e_{\star}^{\phi_k} \star e_{\star}^{-\phi_{k-1}} - m_0^2 (\delta_{n,k} - \delta_{1,k}) e_{\star}^{\phi_1} \star e_{\star}^{-\phi_n}. \tag{19}$$

Note that in the previous expression  $\mu_0 = \mu_n = 0$  and  $k = 1, \dots, n$ . The action from where the nc  $\widetilde{GL}(n)$  affine equations (19) can be derived reads

$$S(\phi_1, \dots, \phi_n) = \sum_{k=1}^n S_{\text{WZNW}_{\star}}(e_{\star}^{\phi_k}) + \frac{k}{2\pi} \int d^2z \left( \sum_{k=1}^{n-1} \mu_k^2 e_{\star}^{\phi_{k+1}} \star e_{\star}^{-\phi_k} + m_0^2 e_{\star}^{\phi_1} \star e_{\star}^{-\phi_n} \right). \tag{20}$$

Among these theories it is found the nc extension of the sine-Gordon model:

$$\begin{aligned} \partial(\bar{\partial}(e_{\star}^{\phi_+}) \star e_{\star}^{-\phi_+}) &= \mu^2 (e_{\star}^{\phi_-} \star e_{\star}^{-\phi_+} - e_{\star}^{\phi_+} \star e_{\star}^{-\phi_-}), \\ \partial(\bar{\partial}(e_{\star}^{\phi_-}) \star e_{\star}^{-\phi_-}) &= \mu^2 (-e_{\star}^{\phi_-} \star e_{\star}^{-\phi_+} + e_{\star}^{\phi_+} \star e_{\star}^{-\phi_-}). \end{aligned} \tag{21}$$

Let us remark that by *Abelian* we refer to a property of the original ordinary commutative theory. On the noncommutative scenario it happens that the zero-grade subgroup  $G_0$  despite it is spanned by the generators of the Cartan subalgebra, turns out to be non-Abelian, i.e., if  $g_1, g_2$  are two elements of the zero-grade subgroup  $G_0$  then  $g_1 \star g_2 \neq g_2 \star g_1$ .

As a resume we constructed in [12] nc extensions of  $GL(n, \mathbb{R})$  Abelian and  $\widetilde{GL}(n, \mathbb{R})$  Abelian affine Toda theories as systems of second order differential equations for  $n$  fields. The nc models differentiate from the commutative case by the presence of an extra field which will not decouple in the equations of motion. The presence of this extra field is due to the introduction of the identity generator in the Cartan subalgebra since the algebra  $\mathcal{SL}(n)$  has been extended to  $\mathcal{GL}(n)$ .

### 3. NC self-dual Chern–Simons system

The Chern–Simons theories in the ordinary commutative space have played a central role in the understanding of different phenomena in planar physics. Recently they have been extended to noncommutative spaces (see, for example, the [28, 31]), where apparently they have proven to be also useful for the description of different phenomena, as for example, the quantum Hall effect [30]. In this context the nc self-dual Chern–Simons system:

$$\bar{D}\Psi = 0, \quad \bar{\partial}A - \partial\bar{A} + [\bar{A}, A]_{\star} = \frac{1}{k} [\Psi^{\dagger}, \Psi]_{\star}, \tag{22}$$

with  $D = \partial + [A, ]_{\star}$  and  $\bar{D} = \bar{\partial} + [\bar{A}, ]_{\star}$  the covariant derivatives, has been considered in different works [27]. Particularly in [9] was considered the nc extension of the Dunne–Jackiw–Pi–Trugenberg (DJPT) [13] (see also [14, 15]) model of a  $U(N)$  Chern–Simons gauge theory coupled to a nonrelativistic complex bosonic matter field on the adjoint representation. The lowest energy soliton solutions of this model satisfy (22) and they are related to the exact solutions of the  $U(N)$  nc chiral model. Through a proposed ansatz nc generalizations of  $U(N)$  Toda and affine Toda theories were constructed [9]. Although in the commutative case this

procedure will lead to the well-known second order differential equations of the conformal Toda or affine Toda theories [13], in the noncommutative scenario the generalization of Toda theories proposed in [9] were expressed as systems of first order differential equations for  $2N - 1$  fields which could not be reduced to coupled second order equations in general.

In this section we would like to relate our nc extensions of the Abelian and Abelian affine Toda models (13), (19) and the proposal presented in [9] for these Toda models. We will see in the following that the nc Leznov–Saveliev equations (8) can be obtained from the nc Chern–Simons self-dual soliton equations (22) expressing the gauge potentials  $A, \bar{A}$  and the matter field  $\Psi$  in a particular way. This gives the possibility of obtaining nc extensions of Toda and affine Toda models as second order differential equations. We will also see that using the equivalence of the nc self-dual equations to the nc chiral model equation it is possible to construct exact solutions of the nc Toda models (13).

From now on we will use the operator formalism. This language is sometimes more convenient since the nonlocality of the star product renders explicitly calculations quite complicated.

The complex coordinates  $z = t + ix$  and  $\bar{z} = t - ix$  satisfy  $[z, \bar{z}] = \theta$ . This suggests that we can represent  $z, \bar{z}$  as creation and annihilation operators  $a = z, a^\dagger = \bar{z}$ . These operators will act on the harmonic-oscillator Fock space  $\mathcal{H}$  with an orthonormal basis  $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$ , for  $n = 0, 1, 2, \dots$ , such that the vacuum is defined as  $a|0\rangle = 0$ . Further

$$a|n\rangle = \sqrt{\theta n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{\theta(n+1)}|n+1\rangle, \quad (23)$$

and the number operator is defined as  $a^\dagger a|n\rangle =: \mathbb{N}|n\rangle = \theta n|n\rangle$ . Any function  $f(t, z, \bar{z})$  on the nc space can be related to an operator-valued function  $\hat{f}(t) \equiv F(t, a, a^\dagger)$  acting on  $\mathcal{H}$  by the use of the Weyl transform [26]. In the operator formalism any field on the nc space becomes an operator on the Hilbert space and the derivatives as well can be represented by operators

$$\partial = \frac{1}{2}(\partial_t - i\partial_x) = \partial_z = -\frac{1}{\theta}[a^\dagger, \quad ], \quad \bar{\partial} = \frac{1}{2}(\partial_t + i\partial_x) = \partial_{\bar{z}} = \frac{1}{\theta}[a, \quad ]. \quad (24)$$

Let us consider the self-dual Chern–Simons system in the operator formalism:

$$\begin{aligned} \bar{D}\hat{\Psi} &= 0, \\ \frac{1}{\theta}[a, \hat{A}] + \frac{1}{\theta}[a^\dagger, \hat{A}] + [\hat{A}, \hat{A}] &= \frac{1}{k}[\hat{\Psi}^\dagger, \hat{\Psi}]. \end{aligned} \quad (25)$$

In the following we will see how the operator version of the nc Leznov–Saveliev equations (8) can also be obtained from (25). For this purpose let us consider that the gauge fields are expressed as

$$\hat{A} = -\frac{1}{\theta}\hat{G}^{-1}[a^\dagger, \hat{G}], \quad \hat{A} = -\hat{A}^\dagger, \quad (26)$$

where  $\hat{G}$  is an element of the complexification of the gauge group  $G$ . Suppose we can decompose  $\hat{G}$  as

$$\hat{G} = \hat{H}\hat{U}, \quad (27)$$

where  $\hat{H}$  is Hermitian and  $\hat{U}$  is unitary. The field strength is then expressed as

$$\hat{F}_{+-} = \frac{1}{\theta}[a, \hat{A}] + \frac{1}{\theta}[a^\dagger, \hat{A}] + [\hat{A}, \hat{A}] = -\frac{1}{\theta^2}U^{-1}H[a, H^{-2}[a^\dagger, H^2]]H^{-1}U \quad (28)$$

and the solution of the self-duality equation  $\bar{D}\hat{\Psi} = 0$  is trivially:

$$\hat{\Psi} = \sqrt{k}\hat{G}^{-1}\hat{\Psi}_0(a)\hat{G}, \quad (29)$$

for any  $\hat{\Psi}_0(a)$ . Inserting this solution in the other self-duality equation (22) yields the equation for  $\hat{H}$ :

$$-\frac{1}{\theta^2}[a, \hat{H}^{-2}[a^\dagger, \hat{H}^2]] = \hat{\Psi}_0^\dagger \hat{H}^{-2} \hat{\Psi}_0 \hat{H}^2 - \hat{H}^{-2} \hat{\Psi}_0 \hat{H}^2 \hat{\Psi}_0^\dagger. \tag{30}$$

What is a second order differential equation for the fields that parameterized  $\hat{H}^2 = \hat{B}$ , i.e. the elements of the zero-grade subgroup. Considering that  $\hat{\Psi}_0 = \epsilon_+$ , i.e. the generator of grade  $\pm 1$  which satisfies  $\epsilon_-^\dagger = \epsilon_+$ , the previous equation is written as

$$-\frac{1}{\theta^2}[a, \hat{B}^{-1}[a^\dagger, \hat{B}]] - [\epsilon_-, \hat{B}^{-1} \epsilon_+ \hat{B}] = 0, \tag{31}$$

which could be the operator version of the nc Leznov–Saveliev equation (8), as can be tested using the Weyl–Moyal map [26] unless the minus sign in front of the second term. This procedure is a nc extension of an alternative way for obtaining the Toda models from the Chern–Simons self-dual equations presented in [13] and it allows us to define the Toda models as second order differential equations, as we will see in the following.

### 3.1. NC self-dual Chern–Simons and Toda field theories

The NCSDCS system (25), as was shown in [12], can be obtained from the nc self-dual Yang–Mills equations in four dimensions through a dimensional reduction process. In this section we will define the Toda field theories (13) starting from this system. Hence this is another example where the Ward Conjecture [24] apparently also works on the nc scenario. Other nc extensions of two-dimensional integrable models have been also derived from the  $D = 4$  nc self-dual Yang–Mills equations [3, 10].

In [9] was proposed for  $U(N)$  the ansatz

$$\hat{A} = \text{diag}(\hat{E}_1, \hat{E}_2, \dots, \hat{E}_N), \quad \hat{\Psi}_{ij} = \delta_{i,j-1} \hat{h}_i \quad i = 1, \dots, N - 1, \tag{32}$$

which after introducing in (25) leads to a system of coupled first order equations for the fields  $\hat{E}_i$  with  $i = 1, \dots, N$  and for the fields  $\hat{h}_i$  with  $i = 1, \dots, N - 1$ :

$$\begin{aligned} \frac{1}{\theta}[a, \hat{h}_i] - \hat{E}_i^\dagger \hat{h}_i + \hat{h}_i \hat{E}_{i+1}^\dagger &= 0, \quad \text{for } i = 1, 2, \dots, N - 1, \\ -\frac{1}{\theta}[a^\dagger, \hat{E}_1^\dagger] + \frac{1}{\theta}[a, \hat{E}_1] + [\hat{E}_1^\dagger, \hat{E}_1] &= -\hat{h}_1 \hat{h}_1^\dagger, \\ -\frac{1}{\theta}[a^\dagger, \hat{E}_i^\dagger] + \frac{1}{\theta}[a, \hat{E}_i] + [\hat{E}_i^\dagger, \hat{E}_i] &= \hat{h}_{i-1}^\dagger \hat{h}_{i-1} - \hat{h}_i \hat{h}_i^\dagger, \quad \text{for } i = 2, \dots, N - 1, \\ -\frac{1}{\theta}[a^\dagger, \hat{E}_N^\dagger] + \frac{1}{\theta}[a, \hat{E}_N] + [\hat{E}_N^\dagger, \hat{E}_N] &= \hat{h}_{N-1}^\dagger \hat{h}_{N-1}, \end{aligned} \tag{33}$$

where we have considered  $k = 1$ . In the ordinary commutative case the corresponding system is reduced to a system of second order differential equations for the fields  $\hat{h}_i$  with  $i = 1, \dots, N - 1$  only. Since the first set of equations in the above system cannot be solved for the fields  $\hat{E}_i$  with  $i = 1, \dots, N$ , the system (33) cannot be reduced to second order differential equations. Looking at (26) we see that  $\hat{A}$  is expressed in terms of first order derivatives

$$\hat{A} = -\frac{1}{\theta} \hat{U}^{-1} \hat{H}^{-1} [a^\dagger, \hat{H}] \hat{U} - \frac{1}{\theta} \hat{U}^{-1} [a^\dagger, \hat{U}]. \tag{34}$$

For  $U(N)$  we can take the constant generators as  $\epsilon_\pm = \sum_{i=1}^{n-1} E_{\pm\alpha_i}$ . As  $\hat{B}$  is an element of the zero-grade subspace it can be represented by a diagonal matrix

$$\hat{B} = \text{diag}(\hat{g}_1, \hat{g}_2, \dots, \hat{g}_N). \tag{35}$$



In order to map the systems (13) and (33) we will consider that the unitary matrix  $U$  is the identity matrix, i.e.  $U = I$ . Now it is possible to choose

$$\hat{\Psi} = \hat{H} \hat{\Psi}_0(a) \hat{H}^{-1}, \quad (36)$$

what will lead to

$$-\frac{1}{\theta^2} [a, \hat{H}^{-2} [a^\dagger, \hat{H}^2]] = -\hat{\Psi}_0 \hat{H}^{-2} \hat{\Psi}_0^\dagger \hat{H}^2 + \hat{H}^{-2} \hat{\Psi}_0^\dagger \hat{H}^2 \hat{\Psi}_0, \quad (37)$$

that can be exactly transformed on the operator version of the nc Leznov–Saveliev equation [19],

$$-\frac{1}{\theta^2} [a, \hat{B}^{-1} [a^\dagger, \hat{B}]] + [\epsilon_-, \hat{B}^{-1} \epsilon_+ \hat{B}] = 0, \quad (38)$$

considering  $\hat{\Psi}_0 = \epsilon_-$ . Now we are in a position of mapping the models. Thus,

$$\hat{\Psi}_{ij} = \delta_{i-1,j} \hat{g}_i^{1/2} \hat{g}_{i-1}^{-1/2} \quad (39)$$

and the relations

$$\begin{aligned} \hat{h}_i^\dagger &= \hat{g}_{i+1}^{1/2} \hat{g}_i^{-1/2}, & \text{for } i = 1, \dots, N-1, \\ \hat{E}_i &= -\frac{1}{\theta} \hat{g}_i^{-1/2} [a^\dagger, \hat{g}_i^{1/2}], & \text{for } i = 1, \dots, N, \end{aligned} \quad (40)$$

are obtained. If we introduce the above relations on the system (33) it is not difficult to see that it reduces to

$$\begin{aligned} -\frac{1}{\theta^2} [a, \hat{g}_1^{-1} [a^\dagger, \hat{g}_1]] &= \hat{g}_1^{-1} \hat{g}_2, \\ -\frac{1}{\theta^2} [a, \hat{g}_i^{-1} [a^\dagger, \hat{g}_i]] &= \hat{g}_i^{-1} \hat{g}_{i+1} - \hat{g}_{i-1}^{-1} \hat{g}_i, & \text{for } i = 2 \dots N-1, \\ -\frac{1}{\theta^2} [a, \hat{g}_N^{-1} [a^\dagger, \hat{g}_N]] &= -\hat{g}_{N-1}^{-1} \hat{g}_N, \end{aligned} \quad (41)$$

which is the operator version of the Toda model (13). The first equations in (33) are trivially satisfied since  $\hat{\Psi}$  was chosen as a solution of these equations. The simplest example of the  $U(N)$  Toda field theories is the nc Liouville model which corresponds to  $N = 2$ . In this case we can write

$$\hat{B} = \begin{pmatrix} \hat{g}_+ & 0 \\ 0 & \hat{g}_- \end{pmatrix}, \quad (42)$$

and the constant generators  $\epsilon_\pm = E_{\pm\alpha}$ . The equations of motion from (41) or equivalently from (31) for this model will be

$$\frac{1}{\theta^2} [a, \hat{g}_+^{-1} [a^\dagger, \hat{g}_+]] = -\hat{g}_+^{-1} \hat{g}_-, \quad \frac{1}{\theta^2} [a, \hat{g}_-^{-1} [a^\dagger, \hat{g}_-]] = \hat{g}_+^{-1} \hat{g}_-. \quad (43)$$

In the same way it is possible to consider the affine models. In [9] the affine ansatz considered was

$$\begin{aligned} \hat{A} &= \text{diag}(\hat{E}_1, \hat{E}_2, \dots, \hat{E}_N), \\ \hat{\Psi}_{ij} &= \delta_{i,j-1} \hat{h}_i, & i = 1, \dots, N-1, \quad \text{except for } \hat{\Psi}_{N1} = \hat{h}_N. \end{aligned} \quad (44)$$

Here again we can established relations analogous to (40) using (34) and (36), but now remembering that  $\epsilon_\pm = \sum_{i=1}^{n-1} E_{\pm\alpha_i}^{(0)} + E_{\mp\psi}^{(\pm 1)}$ . The relations obtained are essentially the relations (40), except the component  $(\hat{\Psi}^\dagger)_{1N} = \hat{h}_{N1}^\dagger = \hat{g}_1^{\frac{1}{2}} \hat{g}_N^{-\frac{1}{2}}$  coming from the extra affine generator.

#### 4. The nc chiral model and the solutions

In the commutative scenario the equivalence of the self-dual Chern–Simons equations and the chiral model equation is significative in the sense that all the solutions of the later have been classified [16]. In [9] was investigated the extension of this relation to a nc spacetime. As we will employ the unition method for the construction of exact solutions of the Toda field theories (41) we will present the relation among the nc chiral model and the nc self-dual Chern–Simons system (25) in the following. Hence, let us consider the new gauge connections,

$$\hat{\mathcal{A}} \equiv \hat{A} - \sqrt{\frac{1}{k}} \hat{\Psi}, \quad \hat{\mathcal{A}}^\dagger \equiv \hat{A}^\dagger + \sqrt{\frac{1}{k}} \hat{\Psi}^\dagger, \tag{45}$$

which satisfy a zero-curvature condition

$$\frac{1}{\theta} [a, \hat{\mathcal{A}}] + \frac{1}{\theta} [a^\dagger, \hat{\mathcal{A}}^\dagger] + [\hat{\mathcal{A}}, \hat{\mathcal{A}}^\dagger] = 0. \tag{46}$$

This means that we can write  $\hat{\mathcal{A}}, \hat{\mathcal{A}}^\dagger$  as pure gauge

$$\hat{\mathcal{A}} = -\frac{1}{\theta} \hat{g}^{-1} [a^\dagger, \hat{g}], \quad \hat{\mathcal{A}}^\dagger = \frac{1}{\theta} \hat{g}^{-1} [a, \hat{g}], \tag{47}$$

for  $\hat{g}$  in some  $U(N)$ . Defining  $\hat{\chi} = \sqrt{\frac{1}{k}} \hat{g} \hat{\Psi} \hat{g}^{-1}$  the nc self-dual Chern–Simons system (22) can be converted into a single equation

$$\frac{1}{\theta} [a, \hat{\chi}] = [\hat{\chi}^\dagger, \hat{\chi}], \tag{48}$$

since

$$\bar{D}\Psi = \sqrt{k} g^{-1} \left( \frac{1}{\theta} [a, \chi] - [\chi^\dagger, \chi] \right) \tag{49}$$

and

$$\frac{1}{\theta} [a, \hat{\mathcal{A}}] + \frac{1}{\theta} [a^\dagger, \hat{\mathcal{A}}^\dagger] + [\hat{\mathcal{A}}, \hat{\mathcal{A}}^\dagger] - \frac{1}{k} [\hat{\Psi}^\dagger, \hat{\Psi}] = g^{-1} \left( \frac{1}{\theta} [a, \chi] - \frac{1}{\theta} [a^\dagger, \chi^\dagger] - 2[\chi^\dagger, \chi] \right) g. \tag{50}$$

Furthermore upon defining

$$\hat{\chi} \equiv -\frac{1}{\theta} \hat{h}^{-1} [a^\dagger, \hat{h}], \quad \hat{\chi}^\dagger \equiv \frac{1}{\theta} \hat{h}^{-1} [a, \hat{h}], \tag{51}$$

for  $\hat{h}$  in the gauge group this equation can be converted into the nc chiral model equation

$$[a, \hat{h}^{-1} [a^\dagger, \hat{h}]] + [a^\dagger, \hat{h}^{-1} [a, \hat{h}]] = 0. \tag{52}$$

In this sense given any solution  $\hat{h}$  of the nc chiral model, or alternatively any solution  $\hat{\chi}$  of (48), we could in principle obtain a solution of the nc self-dual Chern–Simons equations (25). In the ordinary commutative case there is a well-established procedure to construct the solutions of the chiral model equation which have finite energy called the unition method [16]. In [9] was conjectured the extension of this method to the nc plane and was explicitly constructed a specific solution (the simplest) of the nc Liouville model ((33) taking  $N = 2$ ). We would like to use this method to construct exact solutions of our Toda field theories (41) and for this reason we will briefly discuss its details. The main idea is based on one conjecture:

**Conjecture.** *The finite energy solutions  $\hat{h}$  of the nc  $U(N)$  chiral model could be in principle factorized uniquely as a product of unition factors<sup>3</sup>,  $\hat{h} = \hat{h}_0 \prod_{i=1}^m (2p_i - 1)$ , where (a)  $\hat{h}_0$  is a*

<sup>3</sup> In this section for notational simplicity we will write the coordinates as  $z, \bar{z}$  and the derivatives as usual, but they should be understood as in (24). We keep the hats in order to remember that we are working with operators.

constant,  $m \leq N - 1$ , (b) each  $p_i$  is an Hermitian  $N \times N$  projection operator ( $p_i = p_i^\dagger$  and  $p_i = p_i^2$ ) (c) defining  $\hat{h}_j = \hat{h}_0 \prod_{i=1}^j (2p_i - 1)$ , the following linear relations must hold:

$$(1 - p_i) \left( \partial + \frac{1}{2} \hat{h}_{i-1}^{-1} \partial \hat{h}_{i-1} \right) p_i = 0, \quad (1 - p_i) \left( \hat{h}_{i-1}^{-1} \bar{\partial} \hat{h}_{i-1} \right) p_i = 0. \quad (53)$$

This tells us that all the finite energy solutions of nc  $U(2)$  chiral model can be written as  $\hat{h} = \hat{h}_0(2p - 1)$ . That a single unition  $\hat{h} = 2p - 1$ , with  $p$  the Hermitian projection operator satisfying the previous relations is a solution of the nc chiral model equation (52) is very simple to see. Then the next step towards the construction of general solutions involves the composition of unition solutions. The holomorphic projection operator can be written as the projection matrix

$$p = M(M^\dagger M)^{-1} M^\dagger, \quad (54)$$

where  $M = M(z)$  is a rectangular matrix that for  $U(N)$  can be chosen as  $N \times N'$  matrix with  $N' < N$

$$M = \begin{pmatrix} \hat{f}_{11}(z) & \hat{f}_{12}(z) & \cdots & \hat{f}_{1N'}(z) \\ \cdot & \cdot & \cdots & \cdot \\ \hat{f}_{N1}(z) & \hat{f}_{N2}(z) & \cdots & \hat{f}_{NN'}(z) \end{pmatrix}, \quad (55)$$

with  $\hat{f}_{ij}(z)$  being polynomials of  $z$ . Let us remark that these projection operators are related to soliton solutions of the  $\mathbb{CP}^{N-1}$  model [29] and in this case the elements of  $M$  must be polynomial for consistent reasons [9].

At this point let us try to find the solutions of the Toda models (41). So one starts with the  $N$ -dimensional vector

$$u^T = (\hat{f}_1(z), \hat{f}_2(z), \dots, \hat{f}_N(z)), \quad (56)$$

and then defines

$$M_k = (\hat{u}, \partial \hat{u}, \partial^2 \hat{u}, \dots, \partial^{k-1} \hat{u}), \quad (57)$$

which is an  $N \times k$  matrix. On the next step define the projection operators

$$p_k = M_k (M_k^\dagger M_k)^{-1} M_k^\dagger. \quad (58)$$

In this way,

$$\hat{h} = (2p_1 - 1)(2p_2 - 1) \cdots (2p_{N-1} - 1) \quad (59)$$

is a solution of the nc  $U(N)$  chiral model equation. This claim is stated as a theorem in [9] and proven there. The main idea is based on the fact that the vectors  $\hat{u}, \partial \hat{u}, \partial^2 \hat{u}, \dots, \partial^{k-1} \hat{u}$  are considered as linear independent and from these vectors through the Gram–Schmidt process [29] unit vectors  $\hat{e}_i$  are constructed:

$$\hat{e}_i = (1 - p_{i-1}) \partial^{i-1} \hat{u} (\bar{\partial}^{i-1} \hat{u}^\dagger (1 - p_{i-1}) \partial^{i-1} \hat{u})^{-\frac{1}{2}} \quad \text{for } i = 1, \dots, N. \quad (60)$$

Since they span the same space as the vectors  $\hat{u}, \partial \hat{u}, \partial^2 \hat{u}, \dots, \partial^{k-1} \hat{u}$ ,

$$p_k = \tilde{M}_k (\tilde{M}_k^\dagger \tilde{M}_k)^{-1} \tilde{M}_k^\dagger, \quad (61)$$

with  $\tilde{M}_i = (\hat{e}_1, \hat{e}_2, \dots, \hat{e}_i)$ . And since the vectors  $\hat{e}_i$  with  $i = 1, \dots, k - 1$  are orthonormal ( $\hat{e}_i^\dagger \hat{e}_j = \delta_{ij}$ ) it is possible to find a simple expression for  $p_i$ :

$$p_i = \sum_{j=1}^i \hat{e}_j \hat{e}_j^\dagger. \quad (62)$$

Note however that these vectors depend on both coordinates  $z, \bar{z}$ . The unitary matrix  $g = (\hat{e}_1, \hat{e}_2 \cdots \hat{e}_N)$  diagonalizes each  $p_i$

$$g^{-1}p_i g = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 0 \\ 0 & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}, \quad (63)$$

where the first  $i$  entries on the diagonal are 1 and all the others are 0. In the following we will verify that in fact (59) satisfies the nc chiral equation. This is not difficult to see since  $p_i$  is a holomorphic projection operator, i.e.  $p_i \partial p_i = \partial p_i$  for each  $i = 1, \dots, N - 1$ ,

$$\hat{h}^{-1} \partial \hat{h} = \sum_{i=1}^{N-1} (2p_{N-1} - 1) \cdots (2p_{i+1} - 1) \partial p_i (2p_{i+1} - 1) \cdots (2p_{N-1} - 1). \quad (64)$$

By the other side from (63),

$$\partial(\hat{g}^{-1} p_i \hat{g}) = 0, \quad (65)$$

thus

$$\hat{g}^{-1}(\partial p_i) \hat{g} = [\hat{g}^{-1} \partial \hat{g}, \hat{g}^{-1} p_i \hat{g}]. \quad (66)$$

From the Gram–Schmidt procedure it is clear that  $\partial e_i$  is a linear combination of  $e_1, \dots, e_{i+1}$  and by the orthonormality of the basis  $e_i^\dagger \partial e_j = 0$  for all  $i > j + 1$ . Similarly since  $\partial(e_i^\dagger e_j) = 0$ ,  $e_i^\dagger \partial e_j = -(e_j^\dagger \bar{\partial} e_i)^\dagger = 0$  for all  $j > i$ , because  $\bar{\partial} e_i$  is a linear combination of  $e_1, \dots, e_i$ . In this way the matrix  $\hat{g}^{-1} \partial \hat{g}$  has the simple form:

$$\hat{g}^{-1} \partial \hat{g} = \begin{pmatrix} \hat{e}_1^\dagger \partial \hat{e}_1 & & & & 0 \\ \hat{e}_2^\dagger \partial \hat{e}_1 & \hat{e}_2^\dagger \partial \hat{e}_2 & & & \\ & \hat{e}_i^\dagger \partial \hat{e}_{i-1} & \hat{e}_i^\dagger \partial \hat{e}_i & & \\ & & \ddots & \ddots & \\ 0 & & & & \hat{e}_N^\dagger \partial \hat{e}_{N-1} & \hat{e}_N^\dagger \partial \hat{e}_N \end{pmatrix}. \quad (67)$$

This means that  $g^{-1} \partial p_i \hat{g}$  will have only one element different from zero:

$$(g^{-1} \partial p_i \hat{g})_{lm} = \hat{e}_{i+1}^\dagger \partial \hat{e}_i \delta_{l,i+1} \delta_{m,i}. \quad (68)$$

It follows that  $\hat{g}^{-1} [\partial p_i, p_j] \hat{g} = 0$  for  $i < j$  and with this result

$$\hat{h}^{-1} \partial \hat{h} = 2 \sum_{i=1}^{N-1} \partial p_i \quad \text{and} \quad \hat{h}^{-1} \bar{\partial} \hat{h} = -2 \sum_{i=1}^{N-1} \bar{\partial} p_i, \quad (69)$$

since  $\hat{h}$  is unitary and each  $p_i$  is Hermitian. Showing in this way that the chiral equation is satisfied. Then as  $\hat{\chi} = \hat{g} \hat{\Psi} \hat{g}^{-1}$  and from (67), (68), (45) the Toda solution takes the form

$$\begin{aligned} \hat{\Psi}_{ij} = \hat{h}_i^\dagger \delta_{i+1,j} &= [\hat{g}^{-1} \hat{\chi} \hat{g}]_{ij} = -\frac{1}{\theta} \delta_{i+1,j} e_{i+1}^\dagger [a^\dagger, e_i], & i = 1, \dots, N - 1, \\ \hat{A}_{ij} = \hat{E}_i \delta_{i,j} &= \left( \hat{\Psi} - \frac{1}{\theta} \hat{g}^{-1} [a^\dagger, \hat{g}] \right) \delta_{i,j} = -\frac{1}{\theta} \hat{e}_i^\dagger [a^\dagger, \hat{e}_i] \delta_{i,j}, & i = 1, \dots, N. \end{aligned} \quad (70)$$

If we compare this expression with (40) we can see that

$$\begin{aligned}\hat{g}_i^{-\frac{1}{2}} a^\dagger \hat{g}_i^{\frac{1}{2}} &= \hat{e}_i^\dagger a^\dagger \hat{e}_i, & i &= 1, \dots, N \\ \hat{g}_{i+1}^{\frac{1}{2}} \hat{g}_i^{-\frac{1}{2}} &= -\frac{1}{\theta} \hat{e}_{i+1}^\dagger [a^\dagger, \hat{e}_i], & i &= 1, \dots, N-1.\end{aligned}\quad (71)$$

This algorithm allows us to construct exact solutions of the nc Toda field theories (41). In the next section we will see how it explicitly works constructing some simple solutions of the nc Liouville model (43).

#### 4.1. NC Liouville exact solutions

*First solution.* Using the method explained in the previous section, in the work [9] was calculated the simplest exact solution to the nc Liouville model ((33) with  $N = 2$ ). Here we will go one step forward and obtain one solution of (43). Considering the vector  $u^T = (z \ c)$ , with  $c$  being a complex constant, the projection operator  $p = \hat{e}_1 \hat{e}_1^\dagger$  reads,

$$p = \begin{pmatrix} a \frac{1}{\mathbb{N}+|c|^2} a^\dagger & a \frac{1}{\mathbb{N}+|c|^2} \bar{c} \\ c \frac{1}{\mathbb{N}+|c|^2} a^\dagger & |c|^2 \frac{1}{\mathbb{N}+|c|^2} \end{pmatrix}. \quad (72)$$

After computing the orthonormal vector  $e_2$ , the unitary matrix  $\hat{g} = (\hat{e}_1 \ \hat{e}_2)$  is expressed as<sup>4</sup>

$$\hat{g} = \begin{pmatrix} a \sqrt{\frac{1}{\mathbb{N}+|c|^2}} & \sqrt{\frac{|c|^2}{\mathbb{N}+\theta+|c|^2}} \\ c \sqrt{\frac{1}{\mathbb{N}+|c|^2}} & -a^\dagger \frac{1}{\bar{c}} \sqrt{\frac{|c|^2}{\mathbb{N}+\theta+|c|^2}} \end{pmatrix}. \quad (73)$$

Thus

$$\hat{\Psi} = \hat{g}^\dagger \partial p \hat{g} = \begin{pmatrix} 0 & 0 \\ \frac{\sqrt{|c|^2}}{\sqrt{\mathbb{N}+\theta+|c|^2} \sqrt{\mathbb{N}+|c|^2}} & 0 \end{pmatrix}. \quad (74)$$

Using that  $af(\mathbb{N}) = f(\mathbb{N} + \theta)a$ , the gauge potentials  $\hat{A}, \hat{\hat{A}}$  are expressed as

$$\hat{A} = -\frac{1}{\theta} \begin{pmatrix} a^\dagger \left(1 - \frac{\sqrt{\mathbb{N}+|c|^2}}{\sqrt{\mathbb{N}+\theta+|c|^2}}\right) & 0 \\ 0 & a^\dagger \left(1 - \frac{\sqrt{\mathbb{N}+2\theta+|c|^2}}{\sqrt{\mathbb{N}+\theta+|c|^2}}\right) \end{pmatrix}. \quad (75)$$

Taking into account (36) the simplest solution in terms of  $\hat{E}_1, \hat{E}_2, \hat{h}_1$  can be obtained from (70)

$$\begin{aligned}\hat{E}_1 &= \hat{g}_+^{-\frac{1}{2}} [a^\dagger, \hat{g}_+^{\frac{1}{2}}] = a^\dagger \left(1 - \frac{\sqrt{\mathbb{N}+|c|^2}}{\sqrt{\mathbb{N}+\theta+|c|^2}}\right), \\ \hat{E}_2 &= \hat{g}_-^{-\frac{1}{2}} [a^\dagger, \hat{g}_-^{\frac{1}{2}}] = a^\dagger \left(1 - \frac{\sqrt{\mathbb{N}+2\theta+|c|^2}}{\sqrt{\mathbb{N}+\theta+|c|^2}}\right), \\ h_1^\dagger &= \hat{g}_-^{\frac{1}{2}} \hat{g}_+^{-\frac{1}{2}} = \frac{\sqrt{|c|^2}}{\sqrt{\mathbb{N}+\theta+|c|^2} \sqrt{\mathbb{N}+|c|^2}}.\end{aligned}\quad (76)$$

From the above first two equations it is obtained an exact solution of our nc Liouville (43)

$$\hat{g}_+^{\frac{1}{2}} = \alpha(\sqrt{\mathbb{N}+|c|^2}), \quad \hat{g}_-^{\frac{1}{2}} = \beta \left(\frac{1}{\sqrt{\mathbb{N}+\theta+|c|^2}}\right), \quad (77)$$

<sup>4</sup> Unitary in the sense that  $g g^\dagger = 1$ .

where  $\alpha, \beta$  are two constants such that  $\alpha^{-1}\beta = \sqrt{|c|^2}$ . The third condition in (76) fixes the relation among these constants. We can choose  $\alpha = 1$  and  $\beta = \sqrt{|c|^2}$  and the solution is then written as

$$\hat{g}_- = \frac{|c|^2}{\mathbb{N} + \theta + |c|^2}, \quad \hat{g}_+ = \mathbb{N} + |c|^2. \tag{78}$$

In order to study the commutative limit of this solution we apply the Weyl transform [26] that leads to

$$g_- = \frac{|c|^2}{z \star \bar{z} + |c|^2}, \quad g_+ = \bar{z} \star z + |c|^2, \tag{79}$$

or equivalent to

$$g_- = \frac{|c|^2}{\bar{z}z + \theta + |c|^2}, \quad g_+ = \bar{z}z - \theta + |c|^2. \tag{80}$$

Considering a perturbative expansion in  $\theta$ ,

$$\begin{aligned} g_- &= \frac{|c|^2}{\bar{z}z + |c|^2} \left( 1 - \frac{\theta}{\bar{z}z + |c|^2} \right) + O(\theta^2), \\ g_+^{-1} &= \frac{1}{\bar{z}z + |c|^2} \left( 1 + \frac{\theta}{\bar{z}z + |c|^2} \right) + O(\theta^2), \end{aligned} \tag{81}$$

it is not difficult to check that in fact up to first order in  $\theta$  this is a solution of the nc Liouville model [12] with  $g_+ = e^{\phi_+}$  and  $g_- = e^{\phi_-}$  and with  $\varphi_1 = \frac{1}{2}(\phi_+ - \phi_-)$  and  $\varphi_0 = \frac{1}{2}(\phi_+ + \phi_-)$ . This solution in the commutative limit  $\theta \rightarrow 0$  reduces to the well-known Liouville solution:

$$\varphi_1 = \ln \left( \frac{\bar{z}z + |c|^2}{|c|} \right). \tag{82}$$

*Second solution.* As a second example let us now try to find the next simplest solution following the nc extension of the uniton method of [9] outlined in the previous section. For this purpose we will consider  $u^T = (z^2 \ c)$ , from where it is computed the unit vector

$$e_1 = \begin{pmatrix} a^2 \\ c \end{pmatrix} \sqrt{\frac{1}{\mathbb{N}(\mathbb{N} - \theta) + |c|^2}}, \tag{83}$$

and the projection operator

$$p = \begin{pmatrix} a^2 \frac{1}{\mathbb{N}(\mathbb{N} - \theta) + |c|^2} a^{\dagger 2} & a^2 \frac{1}{\mathbb{N}(\mathbb{N} - \theta) + |c|^2} \bar{c} \\ c \frac{1}{\mathbb{N}(\mathbb{N} - \theta) + |c|^2} a^{\dagger 2} & |c|^2 \frac{1}{\mathbb{N}(\mathbb{N} - \theta) + |c|^2} \end{pmatrix}. \tag{84}$$

On the next step we compute the orthonormal vector  $e_2$  using the expression (60) and it reads

$$e_2 = \begin{pmatrix} a \\ -\frac{c}{|c|^2} a^\dagger \mathbb{N} \end{pmatrix} \sqrt{\frac{1}{\mathbb{N}}} \sqrt{\frac{|c|^2}{(\mathbb{N} + \theta)\mathbb{N} + |c|^2}}. \tag{85}$$

By means of the expressions (71) it is computed another solution of the nc Liouville model,

$$\begin{aligned} g_+^{-\frac{1}{2}} a^\dagger g_+^{\frac{1}{2}} &= e_1^\dagger a^\dagger e_1 = a^\dagger \sqrt{\frac{\mathbb{N}(\mathbb{N} - \theta) + |c|^2}{(\mathbb{N} + \theta)\mathbb{N} + |c|^2}}, \\ g_-^{-\frac{1}{2}} a^\dagger g_-^{\frac{1}{2}} &= e_2^\dagger a^\dagger e_2 = a^\dagger \sqrt{\frac{\mathbb{N}}{(\mathbb{N} + \theta)}} \sqrt{\frac{(\mathbb{N} + 2\theta)(\mathbb{N} + \theta) + |c|^2}{(\mathbb{N} + \theta)\mathbb{N} + |c|^2}}. \end{aligned} \tag{86}$$

From where we get that

$$g_+^{\frac{1}{2}} = \alpha \sqrt{\mathbb{N}(\mathbb{N} - \theta) + |c|^2}, \quad g_-^{\frac{1}{2}} = \beta \sqrt{\frac{\mathbb{N}}{(\mathbb{N} + \theta)\mathbb{N} + |c|^2}}. \quad (87)$$

The constants are related through the condition

$$h_1^\dagger = -\frac{1}{\theta} e_2^\dagger [a^\dagger, e_1] = g_-^{\frac{1}{2}} g_+^{-\frac{1}{2}} = \sqrt{\frac{\mathbb{N}}{(\mathbb{N} + \theta)\mathbb{N} + |c|^2}} \sqrt{\frac{4|c|^2}{\mathbb{N}(\mathbb{N} - \theta) + |c|^2}}, \quad (88)$$

from where we obtain that

$$\alpha^{-1} \beta = 2\sqrt{|c|^2}. \quad (89)$$

Choosing  $\alpha = 1$  and  $\beta = 2\sqrt{|c|^2}$ , one solution is then

$$g_- = \frac{4|c|^2 \mathbb{N}}{(\mathbb{N} + \theta)\mathbb{N} + |c|^2}, \quad g_+ = \mathbb{N}(\mathbb{N} - \theta) + |c|^2, \quad (90)$$

that in the commutative limit leads to

$$\varphi_1 = \ln \left( \frac{(\bar{z}z)^2 + |c|^2}{2|c|\sqrt{\bar{z}z}} \right), \quad (91)$$

another known Liouville solution.

*Third solution.* A more general solution that includes the above solutions as particular examples could be computed using the uniton method. Take now the vector as  $u^T = (z^m c)$ . The matrix projector  $p$  in this case is expressed as

$$p = \begin{pmatrix} a^m \frac{1}{\mathbb{N}_m + |c|^2} a^{\dagger m} & a^m \frac{1}{\mathbb{N}_m + |c|^2} \bar{c} \\ c \frac{1}{\mathbb{N}_m + |c|^2} a^{\dagger m} & |c|^2 \frac{1}{\mathbb{N}_m + |c|^2} \end{pmatrix}, \quad (92)$$

where  $\mathbb{N}_m = \mathbb{N}(\mathbb{N} - \theta) \cdots (\mathbb{N} - m\theta + \theta)$ . The matrix  $g$  is equal to

$$g = \begin{pmatrix} a^m \sqrt{\frac{1}{\mathbb{N}_m + |c|^2}} & a^{m-1} \sqrt{\frac{|c|^2}{(\mathbb{N} + \theta)_m + |c|^2}} \frac{1}{\sqrt{\mathbb{N}_{m-1}}} \\ c \sqrt{\frac{1}{\mathbb{N}_m + |c|^2}} & -a^\dagger \frac{1}{c} \sqrt{\frac{|c|^2 \mathbb{N}_{m-1}}{(\mathbb{N} + \theta)_m + |c|^2}} \end{pmatrix}, \quad (93)$$

where  $\mathbb{N}_{m-1} = \mathbb{N}(\mathbb{N} - \theta) \cdots (\mathbb{N} - (m-2)\theta)$  and  $(\mathbb{N} + 1)_m = (\mathbb{N} + \theta)\mathbb{N}(\mathbb{N} - \theta) \cdots (\mathbb{N} - (m-2)\theta)$ . Taking into account (71) it is obtained

$$g_+^{-\frac{1}{2}} a^\dagger g_+^{\frac{1}{2}} = e_1^\dagger a^\dagger e_1 = a^\dagger \sqrt{\frac{\mathbb{N}_m + |c|^2}{(\mathbb{N} + \theta)_m + |c|^2}}, \quad (94)$$

$$g_-^{-\frac{1}{2}} a^\dagger g_-^{\frac{1}{2}} = e_2^\dagger a^\dagger e_2 = a^\dagger \sqrt{\frac{\mathbb{N}_{m-1}}{(\mathbb{N} + \theta)_{m-1}}} \sqrt{\frac{(\mathbb{N} + 2\theta)_m + |c|^2}{(\mathbb{N} + \theta)_m + |c|^2}},$$

from where we get that

$$g_+^{\frac{1}{2}} = \alpha \sqrt{\mathbb{N}_m + |c|^2}, \quad g_-^{\frac{1}{2}} = \beta \sqrt{\frac{\mathbb{N}_{m-1}}{(\mathbb{N} + \theta)_m + |c|^2}}. \quad (95)$$

Once again the constants are related through the condition

$$h_1^\dagger = -\frac{1}{\theta} e_2^\dagger [a^\dagger, e_1] = g_-^{\frac{1}{2}} g_+^{-\frac{1}{2}} = \sqrt{\frac{\mathbb{N}_{m-1}}{(\mathbb{N} + \theta)_m + |c|^2}} \sqrt{\frac{m^2 |c|^2}{\mathbb{N}_m + |c|^2}}, \quad (96)$$

from where we get that

$$\alpha^{-1}\beta = m\sqrt{|c|^2}. \quad (97)$$

Choosing  $\alpha = 1$  and  $\beta = m\sqrt{|c|^2}$ , the solution is then

$$g_- = \frac{m^2|c|^2\mathbb{N}_{m-1}}{(\mathbb{N} + \theta)_m + |c|^2}, \quad g_+ = \mathbb{N}_m + |c|^2. \quad (98)$$

This solution in the commutative limit reduces to another classical Liouville solution:

$$\varphi_1 = \ln \left( \frac{(\bar{z}z)^m + |c|^2}{m|c|\sqrt{(\bar{z}z)^{m-1}}} \right). \quad (99)$$

In this section we have seen how although the nc extension of the unton method is still not proven [9], it can be used to compute exact solutions of the nc Toda models (41). Particularly we have constructed exact solutions of the nc Liouville model (43) and these solutions reduce in the commutative limit to known solutions of the ordinary Liouville model, what in a certain sense gives validity to the method. The construction of exact solutions of other Toda models ( $N > 3$ ) is straightforward, although with much more complicated calculations involved.

## 5. Conclusions

In this paper we have studied in a more detailed way the relation between the nc self-dual Chern–Simons system and the nc Leznov–Saveliev equations. We have seen how from the NCSDCS system it is possible to define the Toda field theories as systems of second order differential equations and still it is possible to construct exact solutions using the nc extension of the unton method proposed in [9]. The solutions explicitly constructed for the nc Liouville model lead to known solutions in the commutative limit, what in a certain way validate the method. Since the NCSDCS system can be obtained from the nc self-dual Yang–Mills equations in four dimensions through a dimensional reduction process [12], the nc Toda field theories could possible have a physical picture inside D-branes systems. Finally we could say that although the complete integrability properties of these theories remain to be investigated, the nc Toda field theories constructed in [12] possess in fact some integrable-like properties: an infinite number of conserved charges,<sup>5</sup> exact solutions, and they are reductions of the nc self-dual Yang–Mills equations in four dimensions that in [25] were shown to be classically integrable.

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<sup>5</sup> Some of these charges can be constructed using the bicomplex formalism (see [1, 12]).



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